Mastering Taylor Problems

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I) Definition : Taylor series is a series that can represent certain types of functions (ln(x+1), Sin(x^2), arctan(x)) in the form of a geometric ((meaning that sum of such series is a number )) series. So, if I ever wanted to know what Sin(0.1) is , I can create a series and find its terms and sum those terms which will give me an approximation of the value of Sin(0.1)

II) When did we use Taylor series in school?

In mechanics, while deriving the differential equations of oscillating rods , we used to say :

Let Sin(Φ) = Φ and Cos(Φ) =

The reason we used to say this IS to make derivation for the Ordinary Differential Equation to be a lot easier by getting rid of the function (cos and sin) and replacing them with polynomials( 1, x ,1294x^4,9x^8….)

So, to recap what I said, Taylor Series is basically a series that is used to represent complicated functions in terms of polynomials

III) How Does Taylor Series Work?

Basically, we can say that Taylor Series is the sum of the derivatives of a certain function. So, if we sum up those derivatives, we will get the original function. But remember that want to find the Taylor series at a certain point.

IV) General Notation :

Let us describe this equation:

We are sometimes asked to find the Taylor series of sin(x) centered at 0. What do they mean by the word CENTRED? do they mean that they want a Taylor series for Sin(0)???

The Answer is NO!! the word centered means that when they find the first second third fourth fifth sixth seventh….nth derivative of sin(x), they want to make sure that those derivatives will always be continuous meaning that I want to find a point and substitute on those derivatives and NOT GET something like 1/infinite or 0/0. And we will find out later that a Taylor series of Sin(x) centered at 0 will always be continuous at 0 because the derivatives of Sin(x) keeps alternating between Sin(x) and Cos(x) and if we are to sub 0 at those derivatives we will get either 0 or 1 and NOT 1/infinite or 0/0

Now you might also ask why it is then the centered part of series looks like this :

(x-a)^n

Because this form reminds us of a geometrical meaning :

Consider the circle of equation : (x-3)^2 + y^2 = 1

This is a circle of center (3,0) and radius. So the circle is CENTRED at x=3 same thing here with the above expression, the \*\*\*graph of each derivative of a function will all be located at a point x=3 and are well defined (meaning graphs are continuous and have values and not (0/0 or infinite/0)\*\*\*

Now What on earth is that weird looking expression?! :

What might be surprising is why is there a factorial there

Well we can recall that a typica polynomial function has the form :

(x – a)^n

And we defined Taylor series as series which represents sum of n derivatives of the above function then finding those derivatives we obtain :

f ‘ (x) = n(x – a)^n-1

f “ (x) = n(n – 1)(x – a)^n-2

f ‘’’(x) = n(n-1)(n-2)(x – a)^n-3

.... f^(n){x} = n(n-1)(n-2)(n-3)…(n – k)(x – a)^n-k

Then we denote n(n-1)(n-2)(n-3)…(n-k) as \*\* n! \*\*

And is the value of the derivative of the function at center x=a

ex: x^2 +7x , its first derivative is 2x + 7 which if using above expression, I’m just subbing n by 1 and if “ a” is 0 then sub 2(0) + 7 , we would get +7 a well defined number and not 1/0

Find the Taylor Series of Sin(x) and deduce Cos(x) centered at 0

F(x) = Sin(x)

Then f ‘(x) = Cos(x) and at center x = 0 we get f ‘(0) = Cos(0) = 1

Now we are going to solve this problem :

First Step : Find first 5 Derivatives of the given function :

f ’(x) = Cos(x)

f “(x) = - Sin(x)

f ”’(x) = - Cos(x)

f ’’” (x) = Sin(x)

f ””’(x) = Cos(x)

Second Step : we substitute x by center a , since the center x= 0 is a center where those derivatives are defined :

f ’(0) = Cos(0) = 1

f “(0) = - Sin(0) = 0

f ”’(0) = - Cos(0) = -1

f ’’” (0) = Sin(0) = 0

f ””’(0) = Cos(0) = 1

Third Step : we apply Taylor’s formula and we must notice a pattern :

Sin(x) =

Sin(x) = 0 + (1)x + (0) + (-1) +(0) +(1)

Sin(x) = x - + …

Observe the pattern :

A) x 🡺x^3🡺x^5 (we had odd powers) thus x^2n+1

B) we have coeff of first term (n=0) as +1 and second term is (n=1) -1 and third term (n=2) +1 thus we have

(-1)^n

C) let’s look at the denominator 1! 🡺 3! 🡺 5! (odd denominator) thus (2n+1)!

Hence combining all together, we get :

Sin(x) =

Now to deduce Cos(x) , we know that the derivative of Sin(x) is Cos(x) so let us find the derivative of the above series with respect to x so treat n as a constant number we thus obtain

Sin(x) = ] = Cos(x)

= ] = Cos(x)

Cos(x) =

Cos(x) =

Memorize these series with terms

A screen shot of a computer

Description automatically generated

Exercise : find the Taylor series of ln(x+1) centered at 0

Recall the definition of “centered” we can notice why the problem wants a Taylor series for ln(x+1) and not ln(x) because the derivatives of ln(x) are not continuous at 0 (we would obtain 0/0 or 1/0) and thus can’t be centered at 0. So, what we can do is find Taylor series for ln(x+1) centered at 0 and if want to estimate some value like ln(1.5) we can convert it to ln(0.5 + 1)

The first 4 derivatives of ln(x+1) are the following

1/x+1 🡺 1/0+1 = 1 = 1!

-1/(x+1)^2 🡺 -1/(0+1)^2 = -1 = -1!

+2/(x+1)^3 🡺 2/(0+1)^3 = 2 = 2!

-(3)(2)/(x+1)^7 🡺 (-3)(2)/(0+1)^7 = (-3)(2) = -3!

Apply Taylor’s Theorem :

ln(x+1) = 0 + (1!) (x)/1! –(1!)(x^2)/2! + (2!)(x^3)/3!

Composite Functions

If we can find a taylor series for a certain function f(x) and if we can find a taylor series for a certain function g(x) , then we can definitely find a composite function f(g(x)) or g(f(x)) example sin(e^x) and this can be achieved by looking at taylor series for sin(x) and substituting instead of x by “ x^n / n! “

and we can thus obtain :

Series Multiplication

We can also find taylor series of 2 different functions f(x) and g(x) so that K(x) = f(x)g(x) example :

Sin(x) = x – + - …

If we divide by x we thus obtain

= 1 - + - …

If you are asked to conclude a general formula for this series we can simply divide the general Taylor series of sin(x) by x and obtain :

=

So you have 2 methods to solve such types of questions either you write the terms and perform the multiplication or division operation or you can write the general series and modify it by performing the above operations. However here we are just demonstrating how we obtained a Taylor representation for sin(x)/x but later you are free to choose which method you want in approximating the \*\*\*error\*\*\* resulting in taylor series

Solve as an exercise : find the Taylor series of

f(x) = tan(x) and f(x) =

hint : use relation tan(x) = sin(x) / cos(x)

Term by Term multiplication

In the example of series multiplication , we multiplied sin(x) by 1/x to obtain sin(x)/x in 2 methods now we present a more complex series multiplication :

(e^x )(cos(x))

an easier but careful calculation method would be to multiply both the general Taylor series of e^x and that of cos(x) and then when asked to estimate the error you then write the first few terms of the founded series

another long but faster method is to apply term by term multiplication :

cos(x) = 1 - + - + …

e^x =1+ x + + + + …

then (e^x)(cos(x)) :

= (1+ x + + + + …)(1 - + - + …)

What I will do is that I want to obtain set of terms with “increase power of x” so by looking at these two series, I can start by multiplying the first term of e^x which is 1 into the first term of cos(x) which is 1 and thus obtain 1 which is lowest power of x “ x^0 “ and it happens that I cant obtain x^0 other than multiplying these 2 terms

Now I want to obtain : (ct number)(x^1) :

I can multiply the 2nd term of e^x into 1st term of cos(x)

And thus (x)(1) = x . And it happens to be that I can get x here only and only by multiplying these terms I mentioned in this case

Now I want to obtain : (some number)(x^2) :

I can multiply the first term of e^x into 2nd term of cos(x):

(1)( -)

and I can also multiply the third term of e^x into the first term of cos (x) : ( )(1) and I can also notice that only these 2 set of multiplication can give me something to the power of x^2 now when we add these 2 results we get

So they cancel out and we get 0 and thus till now we have obtained :

1 + x + 0x^2…

And now you can do the rest on your own

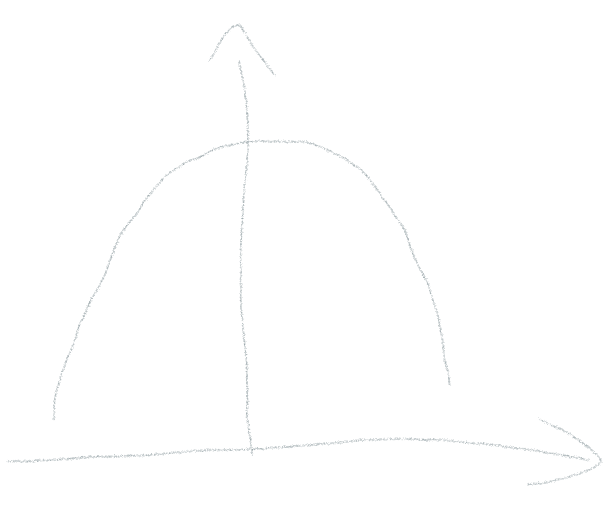
Error Estimation

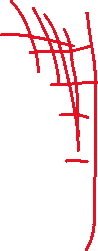
When introducing the definition of taylor series , we mentioned that taylor series is an \*\*\*approximation\*\*\*

of a certain function whether it’s a polynomial or a trigonometric or logarithmic function. In fact , the approximation is not 100% accurate and taylor series is usually done for approximation of values that are so small (x < 1.5)

Why is there an error ???

Well to explain this better suppose I have a parabola and I drew its curve :





Suppose I want to approximate this parabola only by its first term so If a parabola is a quadratic equation (raised to the power of 2 ) then its first derivative is surely going to be a tangent straight line so I have drawn that line at the desired point ( whose abscissa x=0 is the center of the series ) (look at x=0 which is origin you will notice it is indeed the center !!! because everything to the right of the y axis is symmetric to everything on the left of the y axis!!!)

Now look at the red region!! The red region which is the distance between the first derivative of the function to the function itself is a VISULIZATION OF THE ERROR

And surprisingly the error is LARGE, and this makes sense since we approximated the function only by its 1ST DERIVATIVE ONLY

SO, THE MORE WE FIND DERIVATIVE OF THE FUNCTION , THE MORE ACCURATE OUR REPRESENTATION GETS!!! (infact a quadratic equation can be best represented by a combination of cos(something x) and a straight-line y=ax+b )

Look again at the red region if we can “bend” the straight tangent line downward in a “inverse u” shape the error will start to decrease since the distance between that line and the curve are getting closer to each other

So, the correct approximation of a function is the number of terms I chose to approximate plus the remaining terms that I have not used for approximating. Well eventually I can’t go forever approximating a function because we have infinite terms , so we have to stop at some point and thus an error will always exist (and especially when we approximate functions that give irrational numbers because irrational numbers have like infinite random numbers after decimal ex : ln(2) so such types of approximation will always give an error) :

F(x) =Pn(x) + Rn(x)

Pn(x) = nth Taylor Polynomial ( it’s a polynomial composed of the sum of the nth terms that we used to approximate the function F(x))

Rn(x) = Remaining terms

Where the remainder is the 1st term after the last term we used to approximate the function

Rn(x) = (x - a)n+1

Where does c come from ???

we know that :

F(x) =Pn(x) + Rn(x) =

f(a) + (x-a) +….+ (x – a)^n + (x - a)n+1

we can rearrange and thus obtain :

Pn+1(x) = Pn(x) + (x - a)n+1

This means that the maximum obtainable error is :

(x - a)n+1

To clarify I will use this analogy :

If A-B=4 , then the maximum difference between A and B is 4 such that A and B are positive numbers

So, we have founded an UPPER BOUND for the error which represents the maximum obtainable error for estimation :

Rn(x) =MAX( (x - a)n+1)

The maximum error depends on ) since it’s the one that change every time we derive :

MAX(

Thus, maximum error is when ( c IS a ) else an error less than maximum error would be less than x=a ( which is the center of the taylor series ) and thus it would be

x < c < a

things would flip 180 degrees if the function we are approximating is decreasing as x increases such as a hyperbolic function well in that case , everything that I explained would be twisted 180 degrees and thus :

a < c < x

Examples :

1) For what values of x can we replace Sin(x) by x - with an error of magnitude no greater than 3 x 10^-4

Note : No Greater means less than or equal

So here the error is not maximum itself, but the error is a range from 0 to its maximum value

To solve this problem, it’s like we need to move backwards we need to first find Rn(x) which is term after It is infact if you memorize the series of Sin(x) it is :

Thus, the error is ≤

The error is given as 3x10^-4 then :

3x10^-4 ≤

Now it sucks not have calculator during exam good luck

2) Estimate maximum error of Cos(0.4) resulted from the approximation of first 3 terms

2) we know

Cos(0.4) = 1 - + 0<x<0.4

Rn (x) = term after (+ ) Which is ( - )

So, all you have to do is sub 0.4 in that term and we are done !!

3) Estimate Error of = p3(x) with an error less than 10^-8

3) Here you can solve it in 2 methods , either you write Sin(x) as a series and integrate that series then afterwards your write the “FIRST THREE TERMS “ because p3(x) is the sum of the first three terms its what we call TAYLOR’S POLYNIMAL!!!

OR

You can write the terms of sin(x) and then integrate each term

BUT EITHER WAY Rn(x) is the 4th term of the series in both cases and all you have to do when you write Rn(x) is to sub 0.1 and there you have it you found the error and got 20/20 points ☺

Note : DO NOT MIX IN NOTATIONS

For example : R3(x) refers to \*\*4th term\*\* of the series

Note : DO NOT MIX WITH ASET

Different way of solving

CONVERGENCE OF TAYLOR SERIES

IF lim (n🡺 inf) Rn(x) = 0 , this would me that :

F(x) = Pn (x)

But what does this mean???

This means that the more terms we add , we are approaching a more and more accurate form of the function, so this means the error decreases as we terms. If the error was INCREASING as we add terms this would mean that the taylor series DOES NOT represent the function

Example :

Show that Can be represented as f(x) = Sin(x) given : 0<x<z

So, we are asked to prove that the series above is an accurate representation of Sin(x) , now BEWARE we are not asked to find the series of Sin(x) and prove its equal to the series above . Infact the question acknowledges it and asks us to prove that the taylor series of Sin(x) gets more accurate as we increase the terms of the series

So, we will prove lim (n🡺 inf) Rn(x) = 0 :

BY sandwich rule :

So, first of all we would want to write general form of Rn(x)

Rn(x) = (x - a)n+1

We have a problem!!!, we don’t know if the (n+1) derivative of Sin(x) is Cosine or Sine in other words we don’t know if n+1 derivative is 3rd 4th 5th or 99th derivative of Sin(x) so it could be either Sine or Cosine

BUT WAIT!!!!!! In both cases we know that Sine and Cosine are bounded between -1 and 1 thus :

< 1

Applying sandwich rule

0 ≤ ≤

BUT THE ABSOLUTE VALUE OF IS 1 SINCE THE ABS VALUE OF SINE OR COSINE IS 1

So instead of just sub one and now we obtain a basic limit that goes to 0 as n goes to inf so by sandwich rule the Taylor Series Converges and hence we proved what the question wants